LEARNING OBJECTIVES

In this section, you will:

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

7.1 SOLVING TRIGONOMETRIC EQUATIONS WITH IDENTITIES

Figure 1 International passports and travel documents

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the Pythagorean identities, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (see **Table 1**), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Pythagorean Identities		
$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$

Table 1

The second and third identities can be obtained by manipulating the first. The identity $1 + \cot^2\theta = \csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1 + \cot^2\theta = \csc^2\theta$

$$\begin{aligned} 1 + \cot^2\theta &= \left(1 + \frac{\cos^2\theta}{\sin^2\theta}\right) && \text{Rewrite the left side.} \\ &= \left(\frac{\sin^2\theta}{\sin^2\theta}\right) + \left(\frac{\cos^2\theta}{\sin^2\theta}\right) && \text{Write both terms with the common denominator.} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} \\ &= \frac{1}{\sin^2\theta} \\ &= \csc^2\theta \end{aligned}$$

Similarly, $1 + \tan^2\theta = \sec^2\theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1 + \tan^2\theta &= 1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Rewrite left side.} \\ &= \left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Write both terms with the common denominator.} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \end{aligned}$$

The next set of fundamental identities is the set of **even-odd identities**. The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle and determine whether the identity is odd or even. (See **Table 2**).

Even-Odd Identities		
$\tan(-\theta) = -\tan\theta$	$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\cot(-\theta) = -\cot\theta$	$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$

Table 2

Recall that an odd function is one in which $f(-x) = -f(x)$ for all x in the domain of f. The sine function is an odd function because $\sin(-\theta) = -\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\sin\left(\frac{\pi}{2}\right) = 1$$

and

$$\begin{aligned} \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in **Figure 2**.

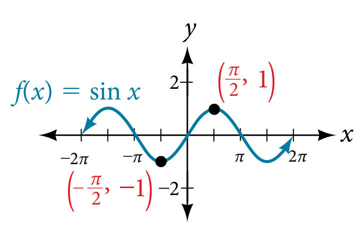

Figure 2 Graph of $y = \sin\theta$

Recall that an even function is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$